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A Numerical Method for Treating Indentation Problems

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SUMMARY
Abel transforms are used to simplify the equations describing the deformation of an elastic medium by a rigid indentor with a circular cross-section. The equations derived constitute a convenient starting point for the numerical solution of such problems. As a particular example the case of a parabolic rough punch is treated.

1. Introduction

The problem of the indentation of an elastic medium by a rigid punch has been considered by several authors, using primarily analytical techniques ([1], [2], [3], [6], [8], [10]). While a number of results have been derived, the problem, at least in some aspects, tends to be a difficult one. Recently one of us (Linz [4], [5]) has used the method of Abel transforms to reduce certain axisymmetric mixed boundary value problems to a form well suited to numerical solution. As we will show in the present paper this technique is quite suitable for the punch problem. The set of equations which we derive is convenient both for the theoretical and numerical treatment; however, our interest is primarily in the numerical approach.

2. Basic Equations

We consider the problem of determining the stresses and deformations in an elastic half-space under the influence of a normal pressure applied by a rigid punch of circular cross-section (Fig. 1). The quantities of interest are the relations between the total penetration δ, the contact radius a, the normal stress σ_z and the shear stress τ_{rz} in the region of contact, the total applied load P, and the shape of the deformed surface. Using cylindrical coordinates the relevant equations are (Abramian [1], Sneddon [7])

$$\sigma_z(r,0) = \int_0^\infty [(1-2\nu)B+\lambda A]\lambda^3 J_0(\lambda r)d\lambda\,, \tag{1}$$

$$\tau_{rz}(r,0) = \int_0^\infty [-2\nu B+\lambda A]\lambda^3 J_1(\lambda r)d\lambda\,, \tag{2}$$

$$u_z(r,0) = -\frac{1}{2\mu}\int_0^\infty [2(1-2\nu)B+\lambda A]\lambda^2 J_0(\lambda r)d\lambda\,, \tag{3}$$

$$u_r(r,0) = -\frac{1}{2\mu}\int_0^\infty [-B+\lambda A]\lambda^2 J_1(\lambda r)d\lambda\,, \tag{4}$$

where μ is the shear modulus, ν Poisson's ratio, and u_r and u_z the displacements in the r and z directions, respectively. A and B are unknown functions of λ to be determined from the boundary conditions

$$\sigma_z(r, 0) = 0\,, \qquad \text{for } r > a\,, \tag{5}$$

$$\tau_{rz}(r, 0) = 0\,, \qquad \text{for } r > a\,, \tag{6}$$

$$u_z(r, 0) = f(r)\,, \qquad \text{for } r \leqq a\,. \tag{7}$$

Here $f(r)$ is a known function, depending on the shape of the punch. It is well known that equations (1)–(7) are not sufficient to describe the problem completely. In particular, the radial displacement u_r, and hence the rest of the quantities, depend on the coefficient of friction between the surfaces. The two extreme cases are (i) the coefficient of friction is zero, or (ii) the coefficient of friction is infinite so that no slip occurs once contact is made. The former is called the smooth punch problem, the latter the rough punch problem.

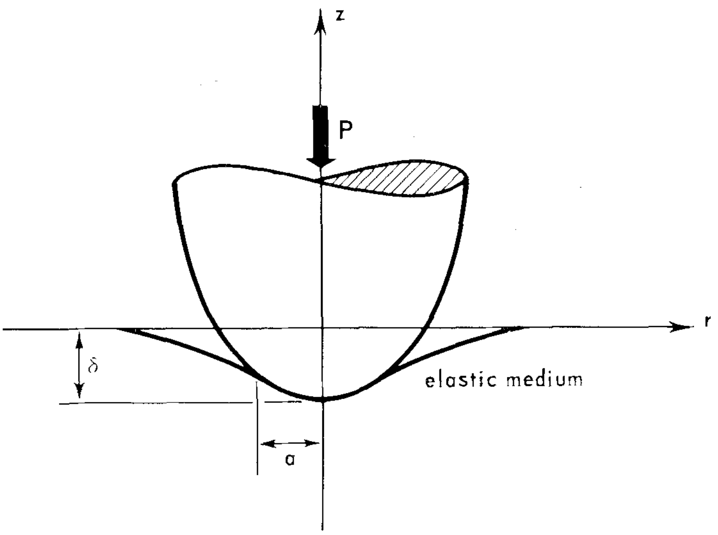

Figure 1. Indentation by a punch.

We simplify the notation by writing $p(r) = \sigma_z(r, 0)$ and $q(r) = \tau_{rz}(r, 0)$. To apply the Abel transform method we introduce the functions $u_1(x)$ and $u_2(x)$ defined by

$$-\pi\mu u_z(r, 0) = \int_0^r (r^2 - x^2)^{-\frac{1}{2}} u_1(x)\,dx\,, \tag{8}$$

$$-\pi\mu u_r(r, 0) = \frac{1}{r}\int_0^r (r^2 - x^2)^{-\frac{1}{2}} u_2(x)\,dx\,. \tag{9}$$

The equations are reduced by inverting (1) and (2) by Hankel's inversion theorem and substituting the resulting expressions for A and B in the inverted forms of (8) and (9). After some manipulations (the details of which may be found in [4]) we arrive at the equations

$$2(1-\nu)\int_x^a \frac{rp(r)}{(r^2-x^2)^{\frac{1}{2}}}\,dr - (1-2\nu)\left[\int_0^a q(r)\,dr - x\int_0^x \frac{q(r)}{(x^2-r^2)^{\frac{1}{2}}}\,dr\right] = u_1(x)\,, \tag{10}$$

$$-(1-2\nu)\int_0^x \frac{rp(r)}{(x^2-r^2)^{\frac{1}{2}}}\,dr + 2(1-\nu)x\int_x^a \frac{q(r)}{(r^2-x^2)^{\frac{1}{2}}}\,dr = u_2(x)\,. \tag{11}$$

Equations (10) and (11) are the basic equations that we will use. As we shall see they form a convenient starting point for analytical and numerical computations.

3. The Smooth Punch

For the smooth punch the shear stress $\tau_{rz}(r, 0)$ vanishes for $r \leqq a$. Thus, with $q(r) = 0$, equations (10) and (11) become simple enough to be solved directly. Omitting the details of the manipulation (which may be found in [4]) we obtain the results for a given contact radius a: Total penetration

$$\delta = -a \int_0^a (a^2 - s^2)^{-\frac{1}{2}} f'(s) ds .$$

Normal stress

$$p(r) = \frac{2\mu}{\pi(1-\nu)} \int_r^a (x^2 - r^2)^{-\frac{1}{2}} \int_0^x (f'(s) + sf''(s))(x^2 - s^2)^{-\frac{1}{2}} ds\, dx .$$

Total load

$$P = -\frac{4\mu}{1-\nu} \int_0^a s^2 (a^2 - s^2)^{-\frac{1}{2}} f'(s) ds .$$

Normal displacement for $r > a$

$$u_z(r, 0) = \frac{2}{\pi} (r^2 - a^2)^{\frac{1}{2}} \int_0^a s(r^2 - s^2)^{-1} (a^2 - s^2)^{-\frac{1}{2}} f(s) ds .$$

These expressions agree with those of Sneddon [8] and Mossakovskii [6].

4. Goodman's Approximation

An approximate method for treating the rough punch was given by Goodman [2]. The approximation is derived by assuming that the contribution to the normal displacement due to the shear stress is negligible. In terms of our formulation this is equivalent to neglecting the terms involving $q(r)$ in equation (10). If this is done, the resulting equations can be solved in a relatively straightforward way, and Goodman's results are obtained. Moreover, looking at it from this point of view, we can see under what circumstances this approximation is likely to yield good results. The neglected term is proportional to $1 - 2\nu$, hence good results may be expected for ν near $\frac{1}{2}$. Comparison of our numerical answers with Goodman's results bears out this prediction (c.f. Table 1).

5. The Rough Punch

The punch problem under the assumption of perfect adhesion in the region of contact has been considered by Mossakovskii [6], Abramian et al. [1], and more recently by Spence [9], [10] and Keer [3], all using essentially analytical techniques. In all cases the final solutions are difficult to use since they involve complicated integrals or integral equations except when the shape of the punch is that of a simple polynomial. The numerical solution of equations (10) and (11) on the other hand appears to be straightforward, except for one difficulty. This difficulty lies in the fact that apparently the radial displacement u_r is generally not known but must be determined as part of the solution. A similar problem occurred in connection with the analytical techniques and Goodman [2] and Mossakovskii [6] argued that the problem must be treated incrementally. That is, at each stage, as the contact radius increases from a to $a + \Delta a$, u_r is unchanged in $0 \leqq r \leqq a$, but $u_r(a + \Delta a)$ is determined by the requirement that the normal and shear stresses at $a + \Delta a$ should remain finite. This incremental formulation can be simulated numerically; such an approach was taken in [4] and is described there in detail. However, after the completion of the work in [4] the results of Spence [9] were brought to our attention, which show that in fact u_r can be determined independently by the use of certain similarity arguments. The results to be described in the next section make use of Spence's work. Nevertheless, the success of the incremental method is of interest since it might be useful in cases where the similarity arguments cannot be used.

6. Numerical Method and Results for a Parabolic Indentor

We assume that the shape of the punch is described by $z=r^2$, in which case

$$f(r)=\delta-r^2\,.$$

Using the results of [9], we find that

$$\delta=\frac{2}{\gamma(\varkappa)}a^2\,,\quad u_r(r,0)=Cr^2\,,$$

where

$$\gamma(\varkappa)=1-0.6931\,\varkappa^2+0.2254\,\varkappa^4+\ldots\,,$$

$$C=-\frac{8\varkappa^2\sqrt{\beta+1}}{3\beta\gamma(\varkappa)}\,,$$

$$\beta=2-4\nu\,,$$

$$\varkappa=\frac{1}{\pi}\ln(1+\beta)\,.$$

Then

$$u_1(x)=-2\mu(\delta-2x^2)\,,$$

$$u_2(x)=-\tfrac{3}{2}\mu\pi Cx^2\,.$$

The numerical method employed consists of splitting the interval $[0, a]$ into N subdivisions by the points $x_0=0$, x_1, x_2, ..., $x_N=a$. The stresses $p(x)$ and $q(x)$ are approximated by their value at the center of each interval so that

$$p_i\simeq p\left(\frac{x_{i-1}+x_i}{2}\right)\,,\quad q_i\simeq q\left(\frac{x_{i-1}+x_i}{2}\right)\,.$$

These approximations are substituted into (10) and (11) and the resulting equations satisfied at $x_1, x_2, \ldots, x_N$, yielding, for $j=1, 2, \ldots, N$, a set of $2N$ linear equations

$$2(1-\nu)\sum_{i=j+1}^{N}m_i(x_j)p_i-(1-2\nu)\sum_{i=1}^{N}(x_i-x_{i-1})q_i-x_j\sum_{i=1}^{j}\bar{n}_i(x_j)q_i=u_1(x_j)\,,\tag{12}$$

$$-(1-2\nu)\sum_{i=1}^{j}\bar{m}_i(x_j)p_i+2(1-\nu)x_j\sum_{i=j+1}^{N}n_i(x_j)q_i=u_2(x_j)\,,\tag{13}$$

where

$$m_i(x)=\int_{x_{i-1}}^{x_i}r(r^2-x^2)^{-\frac{1}{2}}dr=(x_i^2-x^2)^{\frac{1}{2}}-(x_{i-1}^2-x^2)^{\frac{1}{2}}\,,$$

$$\bar{m}_i(x)=\int_{x_{i-1}}^{x_i}r(x^2-r^2)^{-\frac{1}{2}}dr=(x^2-x_{i-1}^2)^{\frac{1}{2}}-(x^2-x_i^2)^{\frac{1}{2}}\,,$$

$$n_i(x)=\int_{x_{i-1}}^{x_i}(r^2-x^2)^{-\frac{1}{2}}dr=\log(x_i+(x_i^2-x^2)^{\frac{1}{2}})-\log(x_{i-1}+(x_{i-1}^2-x^2)^{\frac{1}{2}})\,,$$

$$\bar{n}_i(x) = \int_{x_{i-1}}^{x_i} (x^2 - r^2)^{-\frac{1}{2}} dr = \sin^{-1}(x_i/x) - \sin^{-1}(x_{i-1}/x) .$$

This method is simpler, but somewhat less accurate than the one employed in [4]. To get any reasonable accuracy it is necessary to use unequal spacing, with most of the points concentrated near $x=1$ (where both p and q are not well behaved). For example, the results summarized below were obtained with $N=25$, with the points of subdivision at 0.1 (0.1) 0.7, 0.75, 0.8, 0.85, 0.9, 0.92, 0.94, 0.95, 0.96, 0.97, 0.98, 0.985, 0.99, 0.995, 0.9975, 0.99875, 0.999375, 0.9996875. These were derived experimentally; it is not known what the optimal choice is. Extensive results for this problem were tabulated in [4]; in Table 1 we summarize by giving P as a function of ν (for $\mu=1$ and $a=1$). Also shown is a comparison of Goodman's results with our numerical answers.

TABLE I
Total load P for $\mu=1$ and unit contact radius.

ν	0.1	0.2	0.3	0.4
P	6.71	7.24	7.95	8.96
% error in Goodman's approx., $P=16/3(1-\nu)$	11.6	7.9	4.2	0.7

7. Conclusions

The method presented here is a simple approach to solving certain indentation problems. It has the advantage of avoiding difficult analytical manipulations while leading to equations which are easily solved by numerical techniques. Even more complicated problems than the one presented here can be done. Spence [11] has recently used the method for the flat punch with partial slip (finite, non-zero coefficient of friction) with good results.

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